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A Sum Rule for the Rare Decay Modes of Kaons

PRAKASH CHAND

Colorado State University, Fort Collins, Colorado

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Abstract

An exact sum rule is found for the Ke_4 decay of Kaons and it is modified to account for the possible Charge and Parity-non-conservation.

The purpose of this paper is to derive a new sum rule (1) relating the rare leptonic decays of K-mesons, on the basis of the validity of the $I = \frac{1}{2}$ current rule (Okubo, *et al.*, 1958)†.

$$\Gamma_a(K^+ \to 2\pi^0 + e^+ + \nu_e) + (1+\epsilon)^{-1}\Gamma_b(K_L{}^0 \to \pi^- + \pi^0 + e^+ + \nu_e)$$
$$= \Gamma_c(K^+ \to \pi^+ + \pi^- + e^+ + \nu_e) \qquad (1)$$

where the Γ's are the total decay rates into the channels indicated, ϵ is a possible measure of the deviation from the CP invariance of weak decays. K_L and K_S are the long-lived and short-lived component of neutral K-meson.

Derivations

The differential decay probability $d\sigma$ for K_{e4} (Dalitz, 1964) decay can be written as

$$d\sigma = \sum \frac{(2\pi)^4}{2E_k}|M|^2 \frac{d^3p_e}{(2\pi)^3 2E_e}\frac{d^3p_\nu}{(2\pi)^3 2E_\nu}\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 E_2}\times$$
$$\times\, \delta^4(p_k - p_e - p_\nu - p_1 - p_2) \qquad (2)$$

where the normalization is such that there are $2E$ particles per unit volume E being the energy of the particle. E_k, E_1, E_2, E_e and E_ν are the energies and k, p_1, p_2, p_e and p_ν are the four momenta of the K-meson, pions, electron and neutrino, respectively. The summation is over the polarization of electron and neutrino. M, the matrix element, stands for

$$M = \frac{G}{\sqrt{2}}\left[(f_1 k + f_2 p_1 + f_2 p_1 + f_3 p_2)^\alpha + f_4 \frac{\epsilon^{\alpha\beta\gamma\delta}}{M_e{}^2} k_\beta p_{1\gamma} p_{2\delta}\right] J_\alpha^{e\nu} \qquad (3)$$

where the f's are form factors depending upon the invariant variables

† The $I = \frac{1}{2}$ current rule follows in a well-known way from the assumption that weak currents transform as octet under SU_3 (Cabibbo, N. (1963). *Physical Review Letters*, **10**, 531).

formed from the four momenta of pions and K-meson. G is the weak coupling constant and $J_\alpha^{e\nu}$ is given by $J_\alpha^{e\nu} = \bar{\nu}\gamma_\alpha(1 + \gamma_5)e$, where e and ν are the spinors corresponding to electron and neutrino. In equation (3) the term† $\epsilon^{\alpha\beta\gamma\delta}k_\beta p_{1\gamma}p_{2\delta}M_a^2$ has a M_a^2 in the denominator for dimensional reasons. At the strong vertex of K-decay (strong vertices are treated to all orders) the K-meson will branch to Λ and nucleon and M would be of the order of nucleon mass. Thus the contribution of this term would be very small, and we neglect it. With a little manipulation and using Dirac equation on the leptons, (3) can be written as

$$M = \frac{G}{\sqrt{2}}[f_+(p_1 + p_2) + f_-(p_1 - p_2)]^\alpha J_\alpha^{e\nu} \tag{4}$$

where the mass of electron has been taken to be zero. A straightforward, but tedious, calculation‡ gives the total decay probability

$$\Gamma = \frac{5G^2}{2(2\pi)^6 L_6} \int \Big[f_-^2\{(p_+ . L_+)^2 + 2(p_+^2 L_+^2)\}\left(1 - \frac{4m\pi^2}{p_+^2}\right)$$
$$-6f_+^2\{(p_+ . L_+)^2 - (p_+^2 . L_+^2)\} \sqrt{\left(1 - \frac{4m\pi^2}{p_+^2}\right)} d^4L_+ d^4p_+ \Big] \tag{5}$$

where $L_+ = p_e + p_\nu$.

The expression for total decay probability does not have the cross term $f_+ f_-$. This is the fact which leads to sum rule (1), as is shown below.

The only possible decays of the kind K_{e4} allowed by conservation laws are

$$\begin{aligned} K^+ &\to \pi^+ + \pi^- + e^+ + \nu_e \qquad (X) \\ K^0 &\to \pi^- + \pi^0 + e^+ + \nu_e \qquad (Y) \\ K^+ &\to \pi^0 + \pi^0 + e^+ + \nu_e \qquad (Z) \end{aligned} \tag{6}$$

The $I = \frac{1}{2}$ current rule (Okubo, *et al.*, 1958) says that the pions in the final state must be either $I = 1$ or $I = 0$ state. Whereas both $I = 0$ and $I = 1$ states are possible for the final state of X, Y and Z can have only $I = 0$ and $I = 1$ final states, respectively. Let A^0 and A^1 denote the amplitude corresponding to $I = 0$ and $I = 1$ final state, then the equations (6) will have the following amplitude equation,

$$X = \frac{A^1}{\sqrt{2}} + \frac{A^0}{\sqrt{3}}$$

† I am thankful to Professor Sudarshan for a discussion on this term.

‡ The calculations are simplified by using the identity

$$\frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2} = 4\delta(p_1^2 - m_\pi^2)\,\delta(p^2 - m_\pi^2)\,d^4p_1\,d^4p_2$$
$$= \delta(p_+^2 + p_-^2 - 4m_\pi^2)\,\delta(p_+ p_-)$$

where $p_\pm = p_1 + p_2$ and m_π is the mass of pion. See Dalitz, R. H. (1955). *Physical Review*, 99, 915.

$$Y = +\frac{A^1}{\sqrt{2}} \tag{7}$$

$$Z = -\frac{A^0}{\sqrt{3}}$$

Comparing this with (4) and remembering the symmetry of final state ions, we can write†

$$A^0 = f_+(p_1 + p_2) \tag{8}$$

$$A^1 = f_-(p_1 - p_2)$$

and combining this with the discussion following (5), we get the sum rule

$$\Gamma_a(K^+ \to 2\pi^0 + e^+ + \nu_e) + \Gamma_b(K^0 \to \pi^- + \pi^0 + e^+ + \nu_e)$$
$$= \Gamma_c(K^+ \to \pi^+ + \pi^- + e^+ + \nu_e) \tag{9}$$

It is well known that states with definite lifetime are not K_0 and $\bar{K}_0$ but K_L and K_S. These states are defined by

$$K_S{}^0 = \frac{pK_0 + q\bar{K}_0}{\sqrt{(p^2 + q^2)}}$$

$$K_L{}^0 = \frac{pK^0 - q\bar{K}_0}{\sqrt{(p^2 + q^2)}} \tag{10}$$

In the limiting case of CP invariance

$$p = q = 1 \tag{11}$$

The last equation and $\Delta Q/\Delta S$‡ $= 1$ rule leads us to

$$A(K_S{}^0 \to \pi^- + \pi^0 + e^+ + \nu_e)$$
$$= \frac{p}{\sqrt{(p^2 + q^2)}} A(K^0 \to \pi^- + \pi^0 + e^+ + \nu_e) A(K_L{}^0 \to \pi^- + \pi^0 + e^+ + \nu_e) \tag{12}$$

where the A's are the corresponding amplitudes.

Equations (12) and (9) give us the following two sum rules§

$$\Gamma_a(K^+ \to 2\pi^0 + e^+ + \nu_e) + (1 + \epsilon)^{-1} \Gamma_b(K_L{}^0 \to \pi^- + \pi^0 + e^+ + \nu_e)$$
$$= \Gamma_c(K^+ \to \pi^+ + \pi^- + e^+ + \nu_e) \tag{1}$$

where

$$\epsilon = \frac{p^2 - q^2}{p^2 + q^2}$$

† It is reasonable to assume f_+, f_- constants see Dalitz (1964).

‡ In reference it is shown that $I = \frac{1}{2}$ hadronic current for strangeness changing semi-leptonic decay leads to $\Delta Q/\Delta S = 1$

§ It is well known that K_{e4} decay of short-lived Kaon is at least one order smaller than the K_{e4} decay of long-lived Kaon.

We shall examine the effect of CP invariance and non-invariance on (1). If CP is conserved, (11) leads to

$$\Gamma_a + \Gamma_b^L = \Gamma_c$$

In case CP is not conserved but CPT hold ϵ will give us a possible deviation from CP invariance. It is worthwhile to point out that if CP non-conservation is very small $\epsilon \rightarrow 0$ and we get the sum rule† (1).

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References

Dalitz, R. H. (1964). *Properties of Weak Interaction.* Lectures given at the International School of Physics at Varrenna, 1964. This contains other references on K-decays.

Okubo, S., Marshak, R. E., Sudarshan, E. C. G., Teutsch, W. B. and Weinberg, S. (1958). *Physical Review*, **112**, 665.

† See footnote on previous page.